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Pulse-Position-Modulation Coding as Near-Optimum Utilization of Photon Counting Channel with Bandwidth and Power Constraints

R. G. Lipes
Communications Systems Research Section

We show that the capacity, measured in nats per photon, of a pulse-position modulation (PPM) scheme involving Q uses of a channel that neglects thermal noise and makes binary decisions on the presence of photons (Z-channel) is very close to the optimum capacity for utilizing the Z-channel. For Q in excess of around 100, the differences in capacity are probably insignificant for any practical application. The PPM scheme capacity results as the optimum solution to a communication system design problem.

I. Introduction

In this article we bring together some of the results (Refs. 1-3) on optical communications employing photon counting at the receiver that have been developed over the last couple of years. This will be done by showing that, for communication channels for which thermal noise can be neglected, a pulse-position-modulation (PPM) coding scheme has a capacity over the range of practical interest very close to optimum for reception which detects only the presence or absence of photons. The fact that this PPM scheme is convenient to analyze (Ref. 2) and can be easily utilized in system design considerations enhances the significance of this result.

In the first section, the PPM scheme is explained. A design problem is formulated to obtain the maximum information rate subject to average power and bandwidth constraints. This procedure is equivalent to obtaining the minimum average power subject to information rate and bandwidth constraints, an approach closely related to the work described in Ref. 3.

In the second section, the capacity of the Z-channel (Refs. 1 and 4) is calculated so that a direct, meaningful comparison with the PPM capacity can be made. The Z-channel models the communication system which neglects thermal noise and for which binary decisions are made regarding the reception of photons. The capacity of the Z-channel upper bounds the information exchange per channel use through extension systems, such as the PPM scheme, composed of multiple uses of the Z-channel. Nevertheless, over a range of parameter values achievable by current or projected technology, the PPM capacity, measured in nats per photon, is only slightly inferior to that of the Z-channel.

In the third section, a design problem using PPM is formulated to obtain the maximum information rate subject to peak power and bandwidth constraints. It has been shown in Ref. 5 that the ratio of peak-to-average power must increase exponentially with capacities greater than one nat per photon. Consequently, a different utilization of the PPM system for a

peak power constraint might be expected and is shown to be the case. Presumably, the available technology and particular application will determine whether peak or average power constraints are appropriate for a given situation.

In the fourth section, we discuss some areas that need to be investigated to improve our understanding of optical communication with photon counting reception.

II. Optimized PPM Systems for Certain Optical Communications

In this and the remaining sections we will assume the Z-channel models the physical channel adequately. If no photons are transmitted, none are received as thermal noise is being neglected. If an expected value of λ photons reach the receiver, due to Poisson statistics the probability of none being detected is $e^{-\lambda}$. This Z-channel is depicted in Fig. 1. Suppose a pulse position modulation scheme (Refs. 1 and 2) is used over this channel: in one of Q uses of the channel photons are transmitted with the decoder estimating in which one of the Q slots this transmission occurred. The extended channel model for this system is seen in Fig. 2, where the "0" output indicates all Q slots are estimated to have received zero photons. A practical scheme for mitigating the effect of this "erasure" is to use a Reed-Solomon outer code on the Q-ary channel (Ref. 2). The capacity for the Q-ary PPM channel, achieved when each of the Q codewords is equally likely, is

$$C = (1 - e^{-\lambda}) \log Q \text{ (nats/channel use)} \tag{1}$$

where all rates will be measured in nats unless otherwise stated.

Now suppose we have the design problem of maximizing the information rate for this channel subject to average power and bandwidth constraints. We will take the bandwidth constraint as requiring the duration of one of the Q time slots to be τ. The average power constraint P_{av} (as measured at the receiver to avoid the important but, for this treatment, irrelevant problems of pointing, space loss, etc.) can be expressed as

$$P_{av} = h\nu\lambda/(Q\tau) \tag{2}$$

where h is Planck's constant, ν is the center frequency of the narrowband signal, and λ is the expected number of photons impinging on the receiver.

The information or transmission rate R_T must be less than the channel capacity of Eq. (1) divided by the time for a single channel use:

$$R_T \leqslant (1 - e^{-\lambda}) \log Q/(Q\tau) \equiv (K/\tau)\, \rho \text{ (nats/sec)} \tag{3}$$

where, with the assumed constraints, $K \equiv P_{av}\, \tau/(h\nu)$ is constant and we have introduced ρ, the capacity per photon, a very important parameter for photon communication (Ref. 1). For this channel, ρ has the value

$$\rho = (1 - e^{-\lambda}) \log Q/\lambda \text{ (nats/photon)} \tag{4}$$

Equation (3) shows that in this design problem, maximizing the information rate R_T subject to bandwidth and peak power constraints is equivalent to maximizing ρ. Combining Eqs. (2) and (3) gives

$$\rho = (1 - e^{-KQ}) \log Q/(KQ) \tag{5}$$

which is easily maximized as a function of Q numerically for different values of K.

A related design problem is to minimize the average power (at the receiver) subject to bandwidth and information rate constraints. Using the same notation of the previous design problem, we find the rate constraint implies

$$\lambda \geqslant -\log [1 - \pi Q/\log Q] \tag{6}$$

where the product $R_T\tau \equiv \pi$. Then, the inequality on the average power becomes

$$P_{av} \geqslant h\nu\, [-\log (1 - \pi Q/\log Q)]\, /Q\tau \tag{7}$$

Consequently, minimizing P_{av} is equivalent to maximizing the capacity per photon

$$\rho = \tau Q/[-\log (1 - \pi Q/\log Q)] \tag{8}$$

which is easily done numerically as a function of Q for different values of π. Notice this design problem was addressed and solved in Ref. 3, although the emphasis in that work is somewhat different from that presented here.

These design problems are related in the following obvious way. Assume the same bandwidth constraint is applied for each problem. If the solution maximum rate of the first is used as the constraint value of the second, then the solution minimum average power of the second will be the same value as the constraint average power of the first. Consequently, we can parameterize the optimizing solution by either K or π. The results of numerical calculations are given in Table 1 and plotted in Fig. 3 for values of π that might be expected for

current or projected values of available technologies. Notice that $\pi \equiv R_T\tau$ is exactly the inverse of what is called "bandwidth" expansion in Ref. 5 and is there shown to increase exponentially with ρ for values of greater than one nat/photon. This behavior is quite apparent in Fig. 3.

III. Comparison of PPM Systems With Z-Channel Limit

As stated in the introduction, the capacity of the Z-channel upper bounds the information exchange per channel use through extension systems, such as the PPM scheme, which involve multiple uses of the Z-channel. To compare the PPM scheme with the Z-channel limit, we fixed the value of π as was done in generating Table 1. This parameter π can be viewed as the capacity of the channel per channel use divided by the number of component Z-channel uses. For example, for the Q-ary PPM channel the capacity per channel use is $(1 - e^{-\lambda})$ log Q and there are Q Z-channel uses, so $\pi = [(1 - e^{-\lambda})$ log $Q]/Q$. For the Z-channel itself, π becomes simply the capacity per channel use, since the component Z-channel is used only once. The problem of computing the capacity of the Z-channel subject to an average power constraint is solved in Ref. 4. The solution capacity satisfies

$$C = -q(1 - e^{-\lambda}) - \lambda q e^{-\lambda} - [1 - q(1 - e^{-\lambda})] \log [1 - q(1 - e^{-\lambda})] \quad \text{(nats/channel use)} \tag{9}$$

where λ is the expected number of photons arriving at the receiver, and q is the probability that any photons are transmitted. Maximizing the mutual information subject to the average power constraint requires λ and q to satisfy

$$\log (q^{-1} + e^{-\lambda} - 1) = \lambda(\lambda + 1)/(e^{\lambda} - \lambda - 1) \tag{10}$$

In the "Z-channel" columns of Table 1, we have given the values of q and λ obtained when the capacity of Eq. (9) is fixed at the values of π and the constraint of Eq. (10) is applied. To compare with the PPM channel, notice that Q^{-1} plays the role of q since it corresponds to the probability of any photons being transmitted in a single use of the Z-channel. The capacity per photon for the Z-channel is given by

$$\rho = C/(q\lambda) \tag{11}$$

and upper bounds that for extension channels involving multiple uses of the Z-channel. From the table and Fig. 3, we see the ρ values for PPM are only 5.8 percent and 2.5 percent lower than those for the Z-channel at $Q = 100$ and $Q = 1000$, respectively, with any essential difference disappearing for Q much greater than 1000. Consequently, for Q's as low as 100, the PPM scheme could represent a practical, efficient use of the underlying Z-channel in many applications.

IV. PPM System Optimized With Peak Power Constraint

Consider the design problem of maximizing the information rate for the PPM channel subject to bandwidth and peak power constraints. The peak power seen at the receiver is

$$P_{pk} = h\nu\lambda/\tau \tag{12}$$

since all λ photons arrive in one time slot of duration τ. These constraints fix the expected number of photons, so the information rate satisfying

$$R_T \leqslant (1 - e^{-\lambda}) \log Q/(Q\tau) \tag{13}$$

can be maximized for Q that maximizes $(\log Q)/Q$, or $Q = e$. In this case the capacity per photon given by Eq. (4) is clearly not maximized for fixed λ at $Q = e$. The problem of maximizing information rate subject to bandwidth and peak power constraints is *not* equivalent to maximizing ρ subject to the same constraints. In fact, at the optimizing value of $Q = e$ for the rate maximization, $\rho = (1-e^{-\lambda})/\lambda$, which is upper bounded by 1 nat/photon for all $\lambda \geqslant 0$. For Q-ary PPM, the rates of peak to average power grow as Q. Presumably the available technology and particular application will determine whether the peak or average power constraint is more appropriate, although current laser technology would indicate that peak power constraints are unnecessary for systems with Q less than tens of millions.

V. Areas For Further Study

In this article, we have shown numerically how close the capacity per photon for Q-ary PPM is to the upper bound for optimum use of the Z-channel. For Q in excess of 100 or so, it may be effectively indistinguishable for some applications. Furthermore, the PPM scheme is easy to analyze and we have indicated how it might be utilized in system design considerations. We have compared the PPM system to the Z-channel optimum because both systems make binary decisions regarding the presence of photons at the receiver. It would be very

interesting to determine how good is the practice of making binary decisions in photon counting reception. We know systems with average power constraints that transmit multiple amplitudes in a time slot to communicate more than one bit are more efficient in the nats per photon measure. Work should be carried out to determine how much better they perform, although the bounds of Ref. 5 show no dramatic improvement can be expected.

References

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Table 1. Comparison of parameters of Q-ary PPM and Z channels as a function of capacity per channel use per number of slots

	Q-ary PPM channel			Z-channel		
π	Q	$\lambda\left(\frac{\text{expected no.}}{\text{of photons}}\right)$	$\rho\left(\frac{\text{nats}}{\text{photon}}\right)$	$q\left(\frac{\text{prob. of}}{\text{photons}}\right)$	$\lambda\left(\frac{\text{expected no.}}{\text{of photons}}\right)$	$\rho\left(\frac{\text{nats}}{\text{photon}}\right)$
0.26340	5	1.7054	0.77224	1.3463×10^{-1}	1.4981	1.3053
0.14957	10	1.0486	1.4264	7.3963×10^{-2}	1.1006	1.8374
8.0013×10^{-2}	20	0.76396	2.0947	3.8934×10^{-2}	0.84638	2.4281
3.3726×10^{-2}	50	0.56398	2.9900	1.6279×10^{-2}	0.63671	3.2538
1.7303×10^{-2}	100	0.47116	3.6723	8.3392×10^{-3}	0.53220	3.8987
8.8170×10^{-3}	200	0.40470	4.3573	4.2506×10^{-3}	0.45542	4.5548
3.5930×10^{-3}	500	0.34119	5.2654	1.7348×10^{-3}	0.38116	5.4339
1.8159×10^{-3}	10^3	0.30500	5.9537	8.7805×10^{-4}	0.33874	6.1052
1.8603×10^{-4}	10^4	0.22562	8.2452	9.0413×10^{-5}	0.24617	8.3582
1.8875×10^{-5}	10^5	0.17906	10.541	9.2119×10^{-6}	0.19273	10.631
1.9058×10^{-6}	10^6	0.14844	12.839	9.3316×10^{-7}	0.15815	12.914
1.9190×10^{-7}	10^7	0.12677	15.138	9.4194×10^{-8}	0.13400	15.202
1.9286×10^{-8}	10^8	0.11061	17.438	9.4871×10^{-9}	0.11621	17.493
1.9360×10^{-9}	10^9	9.9154×10^{-2}	19.739	9.5419×10^{-10}	0.10257	19.788
1.9429×10^{-10}	10^{10}	8.8153×10^{-2}	22.040	9.5846×10^{-11}	9.1783×10^{-2}	22.084

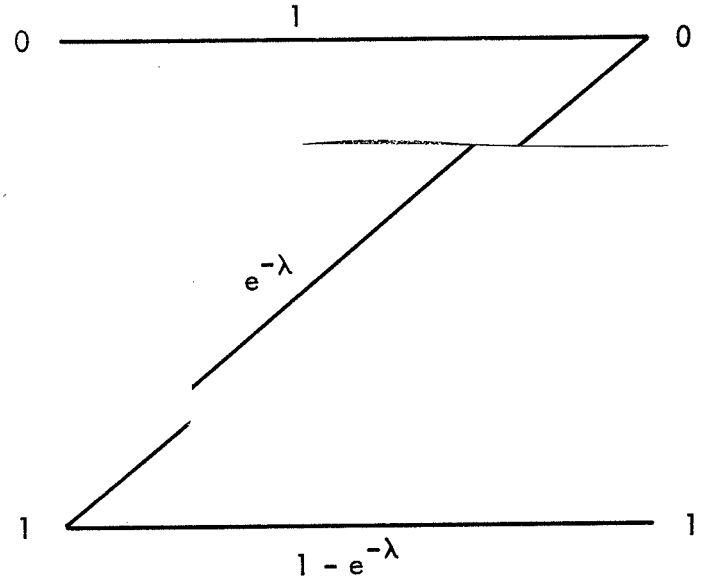

Fig. 1. Z-channel with transition probabilities

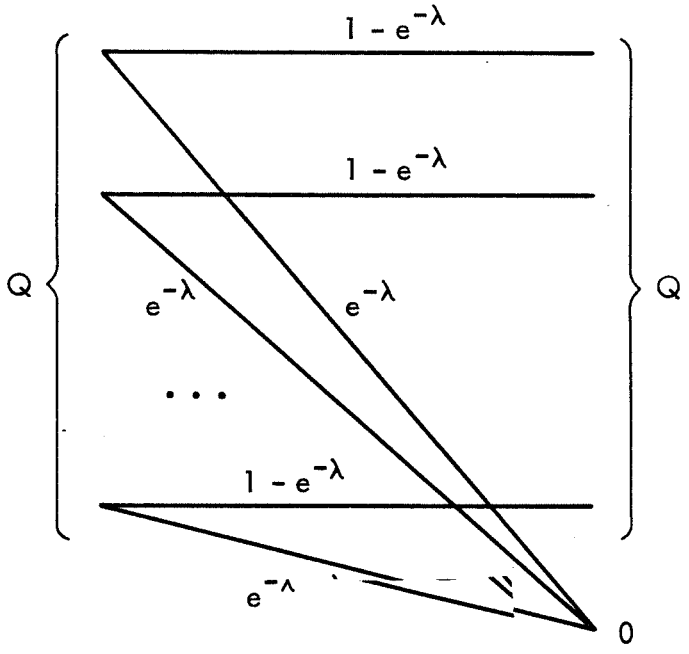

Fig. 2. Q-ary pulse-position-modulation channel with transition probabilities

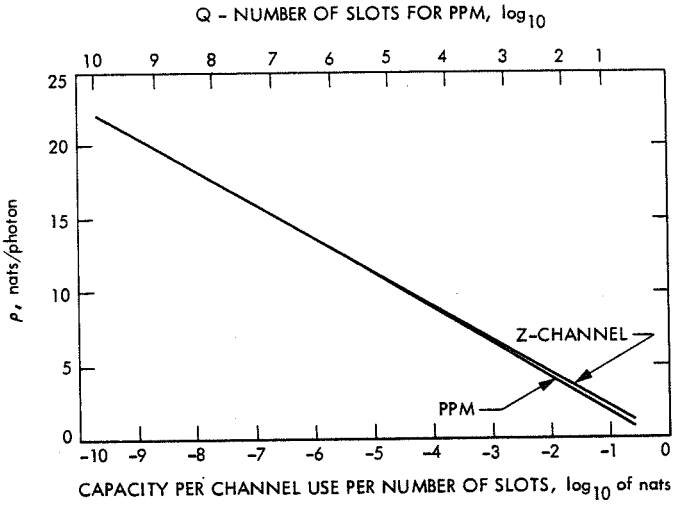

Fig. 3. Variation of ρ for Q-ary PPM and Z-channels with capacity per channel use per number of slots